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Rejecting as in the former case solutions which will transform (1) into a conic, we see that $dx \div d\theta$ is equal to zero if

$$D = B. \tag{17}$$

The eq'n of the cubic will then be

$$y^2 = x^2 \frac{Ax + B}{Cx + D}, \tag{A''}$$

and the straight line joining the extremities of two chords, drawn through the origin at right angles to each other, will pass through a fixed point on the axis of x, the abscissa of which, from eq'n (6) or (15) we find to be

$$x = -\frac{B}{A}. \tag{B''}$$

The fixed point can of course lie at an infinite distance from the origin; the right line joining the extremities of the chords will then be parallel to the axis of x.

Concerning the conics, where the constant angle is 90° and the point from which the chords are to be drawn can be taken anywhere on the curve, the investigation is of such a special interest that it deserves a separate treatise, which we hope to give in another number of this periodical.

INTEGRATION OF THE GENERAL EQUATION OF MOTION.

BY DR. J. MORRISON, NAUTICAL ALMANAC OFFICE, WASH., D. C.

THE first difficulty which the student encounters in reading Gauss's *Theoria Motus Corporum Celestium*, is found in Article 3, where

$$\alpha x + \beta y + r = \gamma$$

is given as the general equation of the conic sections.

The equation is, I believe, due to La Place, who gave a demonstration of it in the Mécanique Céleste, Book II, Chap. III; and therefore, for want of a better name, I shall take the liberty of calling it La Place's Equation to the Conic Sections. I purpose in this brief paper to give a short and easy demonstration of this equation and to discuss it with the view of ascertaining the significance of the constants α, β, γ.

Let us assume the general equations of motion which are

$$\frac{d^2x}{dt^2} = -F\frac{x}{r} \ldots (1), \text{ and } \frac{d^2y}{dt^2} = -F\frac{y}{r} \ldots (2),$$

where F is the force. If F vary as $1 \div r^2$, as is the case in nature, then $F = \mu \div r^2$, where μ is the unit of force at the unit of distance or the absolute force, and (1) and (2) become

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{r^3}\dots(3), \qquad \frac{d^2y}{dt^2} = -\frac{\mu y}{r^3}\dots(4).$$

Multiply (3) by y and (4) by x, subtract the former from the latter and we get

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0.$$

Integrating we have

$$x\frac{dy}{dt} - y\frac{dx}{dt} = \text{a constant}$$
$$= h, \text{ suppose,} \qquad (5)$$

and expressing this in polar coordinates we have the well known equation

$$r^2\frac{d\theta}{dt} = h, \qquad (6)$$

where h is twice the area described by the radius vector in a unit of time.

Again, $x \div r = \cos\theta$;

$$\frac{d}{dt}\left(\frac{x}{r}\right) = -\sin\theta\frac{d\theta}{dt} = -\frac{r\sin\theta}{r^3}r^2\frac{d\theta}{dt} = -\frac{y}{r^3}.h; \text{ whence}$$

$$\frac{y}{r^3} = -\frac{1}{h}.\frac{d}{dt}\left(\frac{x}{r}\right).$$

Similarly

$$\frac{x}{r^3} = \frac{1}{h}.\frac{d}{dt}\left(\frac{y}{r}\right).$$

Substituting in (3) and (4), we have

$$\frac{d^2x}{dt^2} = -\frac{\mu}{h}.\frac{d}{dt}\left(\frac{y}{r}\right),$$
$$\frac{d^2y}{dt^2} = \frac{\mu}{h}.\frac{d}{dt}\left(\frac{x}{r}\right).$$

By the integration of these we have

$$\frac{dx}{dt} = -\frac{\mu}{h}\left(\frac{y}{r} + \beta\right),$$
$$\frac{dy}{dt} = \frac{\mu}{h}\left(\frac{x}{r} + \alpha\right).$$

Substituting in (5) we get

$$\frac{\mu x}{h}\left(\frac{x}{r} + \alpha\right) + \frac{\mu y}{h}\left(\frac{y}{r} + \beta\right) = h,$$

which is easily reduced to

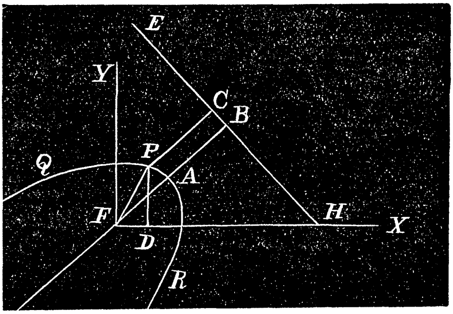
$$\alpha x + \beta y + r = \frac{h^2}{\mu}, \qquad (7)$$

the equation required.

By comparing this with La Place's Equation we see that γ, the absolute term, is necessarily positive and equal to $h^2 \div \mu$; we shall also see presently that γ is an important element in the conic section which the above equation represents. Eliminating r from (7) by means of the eq'n $x^2 + y^2 = r^2$ and reducing we have

$$(1 - \alpha^2)x^2 + (1 - \beta^2)y^2 - 2\alpha\beta xy + 2\alpha\gamma x + 2\beta\gamma y - \gamma^2 = 0, \qquad (8)$$

the general equation to a conic section, the origin being in the present case at the focus. It now remains to ascertain the significance of the constants which enter into this equation, and in order to do this it will be necessary to express the general equation to the conic in the following form.



Let RAQ be a portion of the curve, F the focus and origin, HE the directrix, FX and FY the axes and BF the axis of the curve.

Put $\omega =$ the angle BFX, $p = FB$, $e = PF \div PC$ and x and y the coordin's of P; then we have $FD^2 + PD^2 = PF^2 = e^2\,PC^2$, or $x^2 + y^2 = e^2 PC^2 = e^2(p - x\cos\omega - y\sin\omega)^2$, or

$$(1 - e^2\cos^2\omega)x^2 + (1 - e^2\sin^2\omega)y^2 - 2e^2xy\sin\omega\cos\omega + 2e^2px\cos\omega + 2e^2py\sin\omega - e^2p^2 = 0. \quad (9)$$

By comp'ng (8) and (9) we see that $\alpha^2 = e^2\cos^2\omega$ and $\beta^2 = e^2\sin^2\omega$, whence

$$e = \sqrt{(\alpha^2 + \beta^2)} \ldots (10), \qquad \tan\omega = \beta \div \alpha \ldots (11).$$

If $\omega = 0$, the axis of X coincides with the axis of the curve, but by (11) when $\omega = 0$, $\beta = 0$ and therefore putting $\beta = 0$ in (8) it becomes

$$(1 - \alpha^2)x^2 + y^2 + 2\alpha\beta x - \gamma^2 = 0;$$

and if $x = 0$, then $y = \gamma$, or the simi latus rectum $= \gamma = h^2 \div \mu$, . . . (12) as was shown above.

By comparing the absolute terms of (8) and (9) we have

$$p = \frac{\gamma}{e} = \frac{\gamma}{\sqrt{(\alpha^2 + \beta^2)}}. \qquad (13)$$

The eq. to the directrix EH is, in terms of ω and p, $x\cos\omega + y\sin\omega = p$. Eliminating ω and p by (11) and (13) we get $\alpha x + \beta y = \gamma$ . . . . . (14) for the equation to the directrix in terms of α, β, γ.

The eq'n $\alpha x + \beta y + r = \gamma$ will of course represent an ellipse, a parabola or a hyperbola according as $\sqrt{(\alpha^2 + \beta^2)}$ is less than, equal to, or greater than 1.

In the special case of α and β being both equal to zero, we have $e = 0$, which characterizes a circle whose eq'n will therefore be $r = \gamma = h^2 \div \mu$.

When $\beta = 0$, $\alpha = e$ and the eq'n bec's $ex + r = \gamma = h^2 \div \mu$ and $\alpha = r\cos\theta$,

$$\therefore r = \frac{h^2 \div \mu}{1 + e\cos\theta},$$

the well known polar equation to the conic sections.